Decomposition of bounded degree graphs into C_4-free subgraphs[*]

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Abstract

We prove that every graph with maximum degree Δ admits a partition of its edges into $O(\sqrt{\Delta})$ parts (as $\Delta \to \infty$) none of which contains C_4 as a subgraph. This bound is sharp up to a constant factor. Our proof uses an iterated random colouring procedure.

Keywords: edge-colouring, graph decomposition, graphs of bounded maximum degree, degree Ramsey numbers, bipartite Turán numbers. MSC: 05C15, 05D40, 05C35, 05C55.

1 Introduction

In this paper we consider the following question.

> *Given a graph $G = (V, E)$ with maximum degree Δ, into how few parts can we partition E so that no part has a C_4 subgraph?*

More generally, for any graph H with at least two edges, given $G = (V, E)$ and a map $f : E \to [m]$ for some positive integer m, we call f an *H-free (edge-)colouring of G with m colours* if there is no $i \in [m]$ such that the graph $(V, f^{-1}(i))$ contains H as a subgraph. (Note that this is not necessarily a proper colouring unless H is a two-edge path.) Let $\phi_H(G)$ be the least m such that G admits an H-free colouring with m colours.

Using this notation, the above asks specifically about ϕ_{C_4}, and in answer we show the following.

Theorem 1. *For every graph G with maximum degree Δ, $\phi_{C_4}(G) = O(\sqrt{\Delta})$ as $\Delta \to \infty$.*

In words, every graph with maximum degree Δ admits a partition of its edges (also called a decomposition) into $O(\sqrt{\Delta})$ C_4-free subgraphs.

Let K_n be the complete graph on n vertices. By an upper bound on the size of every colour class in an H-free colouring of $K_{\Delta+1}$, we have that

$$\phi_H(K_{\Delta+1}) \geq \frac{\binom{\Delta+1}{2}}{\mathrm{ex}(\Delta+1, H)} , \tag{1}$$

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where $\mathrm{ex}(n, H)$ as usual denotes the maximum number of edges in an H-free graph on n vertices. Then it follows from an old result of Erdős [3] on the extremal number of C_4 (see [13] for context and more detailed results) that $\phi_{C_4}(K_{\Delta+1}) = \Omega(\sqrt{\Delta})$. This not only shows Theorem 1 to be best possible up to a constant factor, but also foreshadows a central role of the complete graph.

For broader context, Theorem 1 may be understood in terms of the degree Ramsey numbers as first considered in the 1970s by Burr, Erdős and Lovász [1] — they studied these numbers for complete graphs and stars. The more general setting for other graphs was recently revisited in [7]. The question we posed at the beginning is equivalent to finding the multicolour degree Ramsey number of C_4. In [6] it was shown that $\phi_{C_4}(G) = O(\Delta^{9/14})$ for graphs of maximum degree Δ, and the authors asked for the right order of growth. Theorem 1 settles this.

We prove Theorem 1 in Section 3 by using the probabilistic method. In particular, we use an iterated random colouring procedure. At each step of the procedure we identify a collection of large C_4-free colour classes, the removal of which significantly reduces the maximum degree of the graph (see Corollary 7). In the proof, we deliberately make little effort to optimise constants, but we note here that it is possible to obtain a factor less than 45 in Theorem 1 by being more careful at a few points.

Recently, together with Bruce Reed [12], the second author proved that every Δ-regular graph G contains a spanning C_4-free graph with minimum degree $\Omega(\sqrt{\Delta})$. This result has some similarity to our Corollary 7, where instead of looking at the minimum degree of the resulting subgraph, they look at the minimum degree of a given colour class. In a way that is analogous to their work, we essentially reduce our considered problem to the determination of $\mathrm{ex}(\Delta+1, C_4)$. (For us, this is reminiscent of the relationship between independence number and chromatic number found in other extremal colouring problems.)

More generally, we ask the following.

> *For any graph H with at least two edges, is it true that $\phi_H(G) = O(\phi_H(K_{\Delta+1}))$ for every graph G with maximum degree Δ?*

Otherwise stated, we ask if the complete graph on $\Delta + 1$ vertices is essentially the hardest graph to H-free colour among all the graphs with maximum degree Δ.

Trivially, this holds for H a two-edge path. Theorem 1 shows this to be true for $H = C_4$. Using the methods in the proof of Theorem 1, it is possible to confirm this for other bipartite graphs H such as cycles of order twice a prime, or complete bipartite graphs. Moreover, for every $g \geq 4$, we can also edge-colour graphs of maximum degree Δ, each colour class having girth at least g, with an asymptotically tight number of colours. We encourage the reader to consult [12] to see a concrete discussion of how Theorem 4 can be used to upper bound $\phi_H(G)$ for other bipartite graphs H.

Another problem strongly related to our result (via the above displayed question) is to determine $\phi_H(K_n)$. Inequality (1) provides a lower bound on $\phi_H(K_n)$ in terms of $\mathrm{ex}(n, H)$. This prompts us to ask for which graphs H we have $\phi_H(K_n) = O(n^2/\mathrm{ex}(n, H))$ (as $n \to \infty$).

This last statement does not hold if H is not bipartite. On the one hand, Turán's theorem implies that $\mathrm{ex}(n, H) = \Omega(n^2)$. On the other hand, it can be shown in this case that $\phi_H(K_n) = \Omega(\log\log n)$. First observe that $\phi_H(K_n) \geq \phi_{H'}(K_n)$ for any $H \subseteq H'$. Write $|V(H)| = k$ for some fixed $k \geq 3$. The Erdős–Szekeres bound on two-colour Ramsey numbers gives that $R(k, \ell) \leq \binom{k+\ell-2}{k-1} = O(\ell^{k-1})$, so every K_k-free graph of order n has an independent set of size $\Omega(n^{1/(k-1)})$. Let $m = \phi_{K_k}(K_n)$ and let $G_1, \ldots, G_m$ denote the colour classes of a K_k-free colouring of K_n with m colours. Beginning with $V_0 = V$, define V_i to be a maximum independent set of $G_i[V_{i-1}]$ for every $0 < i \leq m$. Then $|V_i| = \Omega(n^{(k-1)^{-i}})$, which implies $m = \Omega(\log\log n)$, as claimed.

Nevertheless, $\phi_H(K_n) = O(n^2/\mathrm{ex}(n,H))$ for some bipartite graphs H such as C_4 [2, 5], C_6 and C_{10} [8].

Bounding or determining the Turán number of bipartite graphs is a central problem in extremal graph theory (see again [13] or, more generally, [4]), so determining for bipartite H the right order of $\phi_H(G)$ in terms of $\Delta(G)$ might be difficult in general.

2 Some probabilistic tools

For our proof we need the following lemmas, the uses of which are covered extensively in [9].

Lemma 2 (Simple Concentration Bound)**.** *Let X be a random variable determined by n trials $T_1, \dots, T_n$ such that for each i, and any two possible sequences of outcomes $t_1, \dots, t_i, \dots, t_n$ and $t_1, \dots, t_i', \dots, t_n$,*

$$|X(t_1, \dots, t_i, \dots, t_n) - X(t_1, \dots, t_i', \dots, t_n)| \le c\,.$$

Then

$$\Pr(|X - \mathbb{E}(X)| > t) \le 2e^{-t^2/(2c^2 n)}\,.$$

Lemma 3 (Lovász Local Lemma)**.** *Consider a set $\mathcal{E}$ of events such that for each $E \in \mathcal{E}$*

- $\Pr(E) \le p < 1$*, and*
- *E is mutually independent from the set of all but at most D of other events.*

If $4pD \le 1$, then with positive probability none of the events in $\mathcal{E}$ occur.

3 Proof of Theorem 1

Before proceeding with the main proof, let us first consider the complete graph $K_{\Delta+1}$. It was shown in the 1970s independently by Chung and Graham [2] and by Irving [5] that, if $\Delta = p^2 + p + 1$ for some prime power p, then $\phi_{C_4}(K_{\Delta+1}) \le p + 1$.

By the density of the primes, it follows easily that

$$\phi_{C_4}(K_{\Delta+1}) \le \lceil 2\sqrt{\Delta}\rceil\,, \tag{2}$$

for all large enough Δ. We later use this in the proof of Theorem 1.

Given a graph $G = (V, E)$, we say that a map $f : V \to [m]$ is 1-*frugal* if it holds for all $i \in [m]$ and $v \in V$ that $|f^{-1}(i) \cap N(v)| \le 1$. We may alternatively view a 1-frugal map as a vertex colouring such that every neighbourhood is *rainbow*. The engine in our proof of Theorem 1 is the following result.

Theorem 4. *Let $G = (V, E)$ be a graph with maximum degree Δ and minimum degree $\delta \ge \log^2 \Delta$ with Δ sufficiently large. For every $\alpha > 16$, there exist $\beta = \beta(\alpha) > 0$, a spanning subgraph H and a (vertex) $(2\lceil\alpha\Delta\rceil)$-colouring χ such that*

- $d_H(v) \ge \beta d_G(v)$ *for every $v \in V$ and*
- *χ is 1-frugal and proper in H.*

Proof. First observe that there exists a spanning bipartite subgraph H_0 such that $d_{H_0}(v) \geq d_G(v)/2$ for every vertex $v \in V$. (Consider H_0 to be a subgraph induced by a maximum edge-cut. This subgraph is clearly bipartite, so let $V = A \cup B$ denote its bipartition. Suppose that $d_{H_0}(v) < d_G(v)/2$ for some $v \in V$. We can assume that $v \in A$. Then the number of edges between $A \setminus \{v\}$ and $B \cup \{v\}$ is strictly larger than the number of edges between A and B, contradicting the maximum edge-cut assumption.) While colouring V, we also construct H as a subgraph of H_0, by sequentially removing edges. The colouring has two consecutive rounds, the first of which colours the vertices of A, the second colours B.

We begin by describing the first round colouring A; this itself has two phases, a probabilistic one followed by a deterministic one.

- Phase I. Colour each vertex $a \in A$ with a colour $\chi_0(a)$ chosen uniformly at random from $[\lceil \alpha\Delta \rceil]$. From χ_0 we obtain a partial colouring χ_1 of A as follows. We uncolour a vertex $a \in A$ if

$$|\{b \in N_{H_0}(a) :\ \exists a' \in N_{H_0}(b) \setminus \{a\}, \chi_0(a') = \chi_0(a)\}| \geq \frac{d_{H_0}(a)}{\sqrt{\alpha}}; \tag{3}$$

  that is, if a certifies that too many of its neighbours have another neighbour in A with colour $\chi_0(a)$. Otherwise, let $\chi_1(a) = \chi_0(a)$ and remove all edges from a to $b \in N_{H_0}(a)$ where b is incident to a' with $a \neq a'$ and $\chi_0(a') = \chi_0(a)$. Let H_1 be the subgraph obtained after removing all these edges. We have ensured that, for any χ_1-coloured $a \in A$ and any $b \in N_{H_1}(a)$, a is the only neighbour of b coloured $\chi_1(a)$.

  We stress that condition (3) is always checked on the initial colouring χ_0 and that all the vertices that are uncoloured lose their colour simultaneously.

- Phase II. Order the uncoloured vertices $a_1, \ldots, a_{s-1}$. For $i = 1, 2, \ldots, s-1$, let $c \in [\lceil \alpha\Delta \rceil]$ be the colour minimising

$$|\{b \in N_{H_i}(a_i) :\ \exists a' \in N_{H_i}(b) \setminus \{a_i\}, \chi_i(a') = c\}| \ .$$

  Delete from H_i all edges $a_i b$ such that there exists $a' \in N_{H_i}(b) \setminus \{a_i\}$ with $\chi_i(a') = c$ and call the resulting subgraph H_{i+1}. Let χ_{i+1} be the partial colouring obtained from χ_i by also assigning a_i the colour c.

First we show that $d_{H_s}(a)$ is large for every $a \in A$.

Claim 5. For every $a \in A$

$$d_{H_s}(a) \geq \left(1 - \frac{1}{\sqrt{\alpha}}\right) d_{H_0}(a) \ .$$

Proof. Note that we only delete edges incident to a at a step in the procedure when a retains its colour. If $a \in A$ retained its colour in the probabilistic phase, we can conclude $d_{H_s}(a) = d_{H_1}(a) \geq (1 - 1/\sqrt{\alpha}) d_{H_0}(a)$, since by (3), conditioned on retaining the colour $\chi_0(a)$, we delete at most $d_{H_0}(a)/\sqrt{\alpha}$ edges incident to a. Otherwise, $a = a_i$ for some $i \in [s-1]$, coloured in the deterministic phase, and since there are at most $d_{H_0}(a_i)\Delta$ edges incident to $N_{H_i}(a_i)$, there exists a colour $c \in [\lceil \alpha\Delta \rceil]$ such that

$$|\{b \in N_{H_i}(a_i) :\ \exists a' \in N_{H_i}(b) \setminus \{a_i\}, \chi_0(a') = c\}| \leq \frac{d_{H_0}(a_i)\Delta}{\lceil \alpha\Delta \rceil} \leq \frac{d_{H_0}(a_i)}{\alpha} \ .$$

Thus $d_{H_s}(a_i) = d_{H_{i+1}}(a_i) \geq (1 - 1/\alpha) d_{H_0}(a_i) \geq (1 - 1/\sqrt{\alpha}) d_{H_0}(a_i)$. □

Claim 6. There exist a spanning subgraph H' and a $\lceil \alpha\Delta \rceil$-colouring χ' of A such that for every $a \in A$

$$d_{H'}(a) \geq \left(1 - \frac{1}{\sqrt{\alpha}}\right) d_{H_0}(a) ,$$

and for every $b \in B$

$$d_{H'}(b) \geq \left(1 - \frac{4}{\sqrt{\alpha}}\right) d_{H_0}(b) ,$$

and $N_{H'}(b)$ is rainbow in χ'.

Proof. Note that the subgraph H_s and colouring χ_s we have constructed are random objects, so it suffices to show that they satisfy the required properties with positive probability (when Δ is large enough). Note that two of the properties are guaranteed by the construction of H_s and χ_s (partly using Claim 5). It only remains to check the degree condition from B.

Let $b \in B$. Observe that the number of coloured neighbours of b under the colouring χ_s is at least the number of coloured neighbours of b under χ_{s-1} (and so on), since in the deterministic phase an edge ab can only be deleted in a step when a is coloured. Thus we can show that $d_{H_s}(b)$ is large by showing that the degree of b in H_1 to the set of vertices coloured by χ_1 is large.

For a given $a \in N_{H_0}(b)$, let E_1 be the event that there exists $a' \in N_{H_0}(b) \setminus \{a\}$ such that $\chi_0(a') = \chi_0(a)$ and let E_2 be the event that a becomes uncoloured (as governed by the condition in (3)). Let Y_b be the random variable that counts the number of vertices $a \in N_{H_0}(b)$ for which E_1 holds. Let Z_b be the random variable that counts the number of vertices $a \in N_{H_0}(b)$ for which E_2 holds but E_1 does not. Notice that these random variables count disjoint sets of vertices. By the observation of the previous paragraph,

$$d_{H_s}(b) \geq d_{H_0}(b) - Y_b - Z_b .$$

We estimate Z_b by studying another random variable. We say that the colour c is *dangerous for a* if

$$|\{b' \in N_{H_0}(a) \setminus \{b\} : \exists a' \in N_{H_0}(b') \setminus \{a\}, \chi_0(a') = c\}| \geq \frac{d_{H_0}(a)}{\sqrt{\alpha}} - 1 .$$

For a given $a \in N_{H_0}(b)$, let E_3 be the event that a receives a dangerous colour. Let Z'_b be the random variable that counts the number of vertices $a \in N_{H_0}(b)$ for which E_3 holds but E_1 does not.

The following observation is important: if a is counted by Z_b it means that a becomes uncoloured and $\chi_0(a)$ is a unique colour within $N_{H_0}(b)$. Then a must have been assigned a dangerous colour since for every vertex $a' \in N_{H_0}(b) \setminus \{a\}$, $\chi_0(a') \neq \chi_0(a)$, and thus a' does not change the number of $b' \in N_{H_0}(a) \setminus \{b\}$ that have colour $\chi_0(a)$ in $N_{H_0}(b') \setminus \{a\}$. Hence $Z_b \leq Z'_b$ and it is enough to verify that not too many vertices receive dangerous colours.

We are going to show that $X_b = Y_b + Z'_b$ is concentrated given any fixed colouring in $A \setminus N_{H_0}(b)$. This, together with an upper bound on the conditional expectation of X_b, suffices to establish an upper bound on X_b that holds unconditionally. During the rest of the proof, we will assume that all the random variables are conditioned to the colouring in $A \setminus N_{H_0}(b)$.

First we deal with the expected value of Y_b. Consider $a \in N_{H_0}(b)$. Observe that at most $d_{H_0}(b) - 1 \leq \Delta$ colours appear in $N_{H_0}(b) \setminus \{a\}$ under the random colouring χ_0. Then the probability that a does not have a unique colour in $N_{H_0}(b)$ is at most $(d_{H_0}(b) - 1)/\lceil \alpha\Delta \rceil \leq 1/\alpha$, and so $\mathbb{E}(Y_b) \leq d_{H_0}(b)/\alpha$.

Second we compute the expected value of Z'_b. Since the maximum degree of H_0 is Δ and a colour is considered dangerous if at least $d_{H_0}(a)/\sqrt{\alpha} - 1$ many vertices $b' \in N_{H_0}(a) \setminus \{b\}$ already have it in $N_{H_0}(b') \setminus \{a\}$, there are at most $d_{H_0}(a)\Delta/(\Delta/\sqrt{\alpha}-1) \le 2\sqrt{\alpha}\Delta$ dangerous colours for a. Thus a receives a dangerous colour with probability at most $2\sqrt{\alpha}\Delta/\lceil\alpha\Delta\rceil \le 2/\sqrt{\alpha}$. So $\mathbb{E}(Z'_b) \le 2d_{H_0}(b)/\sqrt{\alpha}$.

Then

$$\mathbb{E}(X_b) = \mathbb{E}(Y_b) + \mathbb{E}(Z'_b) \le \left(\frac{1}{\alpha} + \frac{2}{\sqrt{\alpha}}\right) d_{H_0}(b) \le \frac{3d_{H_0}(b)}{\sqrt{\alpha}} \ .$$

We can now apply the Simple Concentration Bound to show that X_b is concentrated with polynomially small probability. Note that changing the colour of $a \in N_{H_0}(b)$ can change by at most two the value of X_b:

- it can change by at most two the number of vertices that are unique in their colour class (including a itself), and
- it can change by at most one the number of vertices that receive a dangerous colour and do not satisfy E_1, since the colour classes are prescribed by the colouring given to $A \setminus N_{H_0}(b)$.

Moreover, X_b conditioned on the colouring of $A \setminus N_{H_0}(b)$ is determined by at most $d_{H_0}(b)$ many different trials. By the Simple Concentration Bound with the choices $c = 2$ and $n = d_{H_0}(b)$, we have that X_b conditioned to any colouring in $A \setminus N_{H_0}(b)$ is unlikely to be large:

$$\Pr\left(X_b \ge \frac{4d_{H_0}(b)}{\sqrt{\alpha}}\right) \le \Pr\left(X_b - \mathbb{E}(X_b) \ge \frac{d_{H_0}(b)}{\sqrt{\alpha}}\right) \le 2\exp\left(-\frac{d_{H_0}^2(b)}{8\alpha \cdot d_{H_0}(b)}\right) = e^{-\Omega(d_{H_0}(b))} = o(\Delta^{-6}) \ .$$

In the last equality we used that $d_{H_0}(b) = \Omega(\log^2 \Delta)$. Thus the previous inequality also holds for the unconditioned random variable X_b.

Observe that X_b depends on the vertices at distance at most 3 from b; the fact that $a \in N_{H_0}(b)$ retains its colour depends only on the colours assigned to vertices at distance 2 from a. Thus every event corresponding to X_b is mutually independent from the set of events corresponding to $X_{b'}$ with b' at distance more than 6 from b, the Lovász Local Lemma yields that with positive probability $X_b \le 4d_{H_0}(b)/\sqrt{\alpha}$ for every $b \in B$. This completes the proof of the claim. □

In the second round, we can apply the same argument to colour the vertices of B using the subgraph H'. By Claim 6 and recalling that $\alpha > 16$, this graph has minimum degree at least $(1-4/\sqrt{\alpha})\delta(H_0) = \Omega(\log^2 \Delta)$ and maximum degree at most Δ. So we can apply the same procedure (and claims) to colour B with a new set of $\lceil\alpha\Delta\rceil$ colours. Combined with the colouring χ' of A, in this way we obtain a subgraph $H \subseteq H'$ and a $(2\lceil\alpha\Delta\rceil)$-colouring χ of V such that

- for every $v \in V$
$$d_H(v) \ge \left(1 - \frac{4}{\sqrt{\alpha}}\right)^2 d_{H_0}(v) \ge \left(1 - \frac{4}{\sqrt{\alpha}}\right)^2 \frac{d_G(v)}{2} \ , \text{ and}$$
- χ is a 1-frugal proper colouring of H.

This proves the theorem with the choice $\beta = \frac{1}{2}\left(1 - 4/\sqrt{\alpha}\right)^2$. □

Corollary 7. *Let G be a graph with maximum degree Δ and minimum degree $\delta \ge \log^2 \Delta$ with Δ sufficiently large. For every $\alpha > 16$, there exist $\beta = \beta(\alpha) > 0$ and $\ell \le \lceil 2\sqrt{2\lceil\alpha\Delta\rceil}\rceil$ many C_4-free disjoint spanning subgraphs $G_1, \dots, G_\ell$ such that for all $v \in V$*

$$\sum_{i=1}^{\ell} d_{G_i}(v) \ge \beta d_G(v) \ .$$

Proof. We use the subgraph H and the colouring χ guaranteed by Theorem 4 to find many C_4-free spanning subgraphs. By (2), for any sufficiently large t there exists a decomposition of K_t into C_4-free subgraphs $\mathcal{G}_1, \ldots, \mathcal{G}_{\lceil 2\sqrt{t} \rceil}$. Consider $t = 2\lceil \alpha\Delta \rceil$ and for any $i \in [\lceil 2\sqrt{t} \rceil]$ construct G_i as follows:

- $V(G_i) = V(G)$ and
- $uv \in E(G_i)$ if and only if $uv \in E(H)$ and $\chi(u)\chi(v) \in E(\mathcal{G}_i)$.

These subgraphs G_i are disjoint and, since H contains no monochromatic edge, each edge of H appears in exactly one subgraph G_i. So the minimum degree condition for H implies the minimum degree sum condition demanded here. Moreover, each G_i is C_4-free: by χ being 1-frugal and proper, all 4-cycles in H are rainbow; and if G_i contains such a 4-cycle C, then the colours $\chi(C)$ form a 4-cycle in $\mathcal{G}_i$. $\square$

Besides the above, we need the following bound on arboricity by degeneracy (which follows, for instance, from the folkloric Proposition 3.1 of [11] combined with an old result of Nash-Williams [10]).

Lemma 8. *Let $G = (V, E)$ be a graph with an ordering $(v_1, \ldots, v_n)$ of V which satisfies that $|N(v_i) \cap \{v_{i+1}, \ldots, v_n\}| \leq k$ for all $i \in [n]$. Then E can be partitioned into k parts such that no part contains a cycle of G.*

Proof of Theorem 1. Let $G = (V, E)$ be a graph with maximum degree Δ and fix $\alpha > 16$. We perform the following procedure.

1. Let $\tilde{G}^0 = G$ and $G' = (V, \emptyset)$.
2. Start with $i = 0$ and repeat the following until $i = \tau$, where τ is the smallest such that $\Delta(\tilde{G}^\tau) \leq \log^2 \Delta$:
   (a) obtain G^i from $\tilde{G}^i$ by successively removing all vertices of degree less than $\log^2 \Delta$, and adding all of their incident edges to G';
   (b) apply Corollary 7 to G^i to obtain the disjoint C_4-free subgraphs $G_1^i, G_2^i, \ldots, G^i_{\lceil 2\sqrt{2\lceil \alpha\Delta(G^i)\rceil}\rceil}$;
   (c) set $\tilde{G}^{i+1} = (V(G^i), E(G^i) \setminus \bigcup_j E(G_j^i))$ and then increment i.
3. Add all edges of $\tilde{G}^\tau$ to G'.

We can always apply Corollary 7 at each iteration since in Step 2(a) we forced the minimum degree of G^i to be at least $\log^2 \Delta \geq \log^2 \Delta(G^i)$.

Let us see that the maximum degree $\Delta(G^{i+1})$ is significantly smaller than $\Delta(G^i)$. By Corollary 7, the removal of C_4-free subgraphs at iteration i removes at least $\beta d_{G^i}(v)$ edges incident to $v \in V$. Thus

$$\Delta(G^{i+1}) \leq \Delta(\tilde{G}^{i+1}) \leq (1-\beta)\Delta(G^i) \leq (1-\beta)^i \Delta \,. \tag{4}$$

This implies that the procedure is guaranteed to stop after $\tau = O(\log \Delta)$ iterations.

Step 2(b) of each iteration generates a number of disjoint spanning C_4-free subgraphs, each of which we give a new colour. During the ith iteration we produce $\lceil 2\sqrt{2\lceil \alpha\Delta(G^i)\rceil}\rceil < 2\sqrt{2\alpha\Delta(G^i)} + 4$ such subgraphs, so by (4) and the bound on the number τ of iterations we produce at most

$$O(\log \Delta) + 2\sqrt{2\alpha\Delta} + 2\sqrt{2\alpha(1-\beta)\Delta} + 2\sqrt{2\alpha(1-\beta)^2\Delta} + \cdots = \frac{2\sqrt{2\alpha}}{1-\sqrt{1-\beta}} \cdot \sqrt{\Delta} + O(\log \Delta) \tag{5}$$

C_4-free subgraphs throughout all iterations.

It only remains to upper bound the number of colours needed in the remainder graph G'. By construction, G' admits a degeneracy ordering satisfying the hypothesis of Lemma 8 for $k = \log^2 \Delta$. Thus we can partition its edges into at most $\log^2 \Delta$ acyclic (and thus C_4-free) subgraphs. By (5) we obtain a partition of E into $O(\sqrt{\Delta})$ C_4-free subgraphs in total. This completes the proof of the theorem. $\square$

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